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*Acta Cryst.* (1993). A**49**, 806–817

# The Classification of Face-Transitive Periodic Three-Dimensional Tilings

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(*Received* 16 *January* 1992; *accepted* 7 *April* 1993)

## Abstract

It has long been known that there exists an infinite number of types of *tile*-transitive periodic three-dimensional tilings. Here, it is shown that, by contrast, the number of types of *face*-transitive periodic three-dimensional tilings is finite. The method of Delaney symbols and the properties of the 219 isomorphism classes of crystallographic space groups are used to find exactly 88 *equivariant* types that fall into seven topological families.

## 0. Introduction

Consider the three-dimensional Euclidean space $\mathbb{E}^3$. A point-set $P \subset \mathbb{E}^3$, together with a finite family $F(P)$ of *faces* $f \subset P$, is called a *topological polyhedron* if it satisfies the following conditions:

($P1$) the set $P$ is homeomorphic to the unit ball $B_3 := \{x \in \mathbb{R}^3 \,\|\, |x| \leq 1\}$;

($P2$) the union of faces covers the boundary of $P$, *i.e.* $\bigcup_{f \in F(P)} f = \partial P$;

($P3$) each face $f \in F(P)$ is homeomorphic to the disc $D := \{x \in \mathbb{R}^2 \,\|\, |x| \leq 1\}$;

($P4$) the intersection of any number of distinct faces is either empty, a point (called a *vertex*) or an arc (called an *edge*), that is, homeomorphic to the interval $I := \{x \in \mathbb{R} \,\|\, |x| \leq 1\}$;

($P5$) each face contains at least three vertices.

Note that it follows from these conditions that – dually to ($P5$) – each vertex is contained in at least three edges.

A system $\mathcal{T} = \{P_1, P_2, P_3, \ldots\}$ of topological polyhedra (called *tiles*) is called a (face-to-face) *tiling* of $\mathbb{E}^3$, or *three-dimensional tiling*, if it satisfies the following conditions:

($T1$) the tiling covers space, *i.e.* $\bigcup_{P \in \mathcal{T}} P = \mathbb{E}^3$;

($T2$) the intersection of any two distinct tiles $P$ and $P'$ is either empty, a common vertex, a common edge or a common face. The tiles, faces, edges and vertices associated with $\mathcal{T}$ are called the *constituents* of $\mathcal{T}$ (of dimension 3, 2, 1 and 0, respectively).

A three-dimensional tiling $\mathcal{T}$ is called *periodic* if there exists a discrete group $\Gamma$ of isometries of $\mathbb{E}^3$, containing three linearly independent translations, *i.e.* a *crystallographic space group*, such that $\mathcal{T} = \gamma\mathcal{T} := \{\gamma P \,|\, P \in \mathcal{T}\}$ for all $\gamma \in \Gamma$ (with $\gamma P := \{\gamma p \,|\, p \in P\}$, of course) and $F(\gamma P) = \gamma F(P)$ for all $P \in \mathcal{T}$ and $\gamma \in \Gamma$. In this case, the pair $(\mathcal{T}, \Gamma)$ is an *equivariant tiling* as defined by Dress (1984, 1987).

More specifically, if – as above – $\mathcal{T}$ is a three-dimensional tiling and if $\Gamma$ is a crystallographic group, then we call the pair $(\mathcal{T}, \Gamma)$ an *equivariant three-dimensional tiling*.‡ Two equivariant three-dimensional tilings $(\mathcal{T}, \Gamma)$ and $(\mathcal{T}', \Gamma')$ are called *topologically equivalent* (or are described as being in the same *topological family*) if there exists a homeomorphism $\varphi: \mathbb{E}^3 \to \mathbb{E}^3$ that maps the tiles of one tiling onto the tiles of the other, *i.e.* if $\varphi\mathcal{T} = \mathcal{T}'$. If, additionally, $\Gamma' = \varphi\Gamma\varphi^{-1}$ holds, then the two are called *equivariantly equivalent*.

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‡ Even more specifically, such a pair should be called an equivariant *Euclidean* three-dimensional tiling, where the term Euclidean indicates that the group $\Gamma$ is supposed to consist of isometries with respect to the Euclidean metric of $\mathbb{E}^3$ (as opposed to arbitrary groups of homeomorphisms of $\mathbb{E}^3$).




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Note that, according to our definition, the symmetry group $\Gamma$ is not necessarily identical with the *full* group of isometries of $\mathbb{E}^3$ that map the tiling onto itself. If one wants the two to coincide, then this can always be achieved by allowing the tiles to carry additional structure, for example, the so-called 'marks', that breaks superfluous symmetries [see Grünbaum & Shephard (1987, p. 269)]. 'Equivariantly equivalent' tilings are sometimes also called *homeomeric* or, more precisely, *marked-homeomeric* [see Grünbaum & Shephard (1981*b*)].

Let $(\mathcal{T},\Gamma)$ be an equivariant three-dimensional tiling. Two tiles $P,P' \in \mathcal{T}$ are called *equivalent* (or *Γ-equivalent*) if there exists a *symmetry* $\gamma \in \Gamma$ with $P = \gamma P'$. The *equivalence* of faces, edges and vertices of tiles is defined in the same way. An equivariant three-dimensional tiling $(\mathcal{T},\Gamma)$ is called *tile-*, *face-*, *edge-* or *vertex-transitive* if all its tiles, faces, edges or vertices, respectively, are equivalent. If the number of equivalence classes is $k \in \mathbb{N}$, then the tiling is also called *tile-*, *face-*, *edge-* or *vertex-k-transitive*.

A simple construction (see Heesch, 1934) shows that the number of equivariant classes of *tile*-transitive three-dimensional tilings is infinite. In this paper, we show that, by contrast, there exist only 88 equivariant classes of *face*-transitive three-dimensional tilings. These tilings fall into seven topological families: the cube (or rhombohedron), tetrahedron (or, rather, sphenoid), rhombic dodecahedron, octahedron (or, rather, dipyramid), special rhombohedron, covered rhombohedron and octahedron–tetrahedron tilings (see §4).

In §1 of this paper, we introduce the main tool used in our investigations, the theory of *Delaney symbols*, applied here to periodic three-dimensional tilings. §2 contains the proof of the fact that the number of equivariant types of face-transitive three-dimensional tilings is finite. We then discuss, in §3, how to generate by computer, and then single out 'by hand', the 88 classes mentioned above. Finally, in §4, we briefly discuss the results and some examples.

Our approach can be used to classify the face-transitive three-dimensional tilings of all eight three-dimensional geometries, discussed by Thurston (1980) and Scott (1983). Let us also remark that the classification of *edge*-transitive Euclidean three-dimensional tilings can easily be obtained by dualizing the 88 *face*-transitive tilings. This will be discussed in a future paper.

## 1. Delaney symbols

The following method of 'encoding' periodic tilings in the forms of certain colored graphs called Delaney symbols was inspired by Delaney (1980), suggested and introduced by Dress (1984), first applied by Dress & Scharlau (1984) and worked out in more detail in a number of papers including those of Dress (1987) and Dress & Huson (1987). Based on the theory of Delaney symbols, several computer programs have been developed to solve classification problems in tiling theory [see Delgado Friedrichs, Huson & Zamorzaeva (1992), Franz & Huson (1992) and Huson (1993)].

To start with, let $(\mathcal{T},\Gamma)$ be an equivariant three-dimensional tiling (see Fig. 1) and consider the set of (maximal) flags associated with $\mathcal{T}$, that is, the set $\mathcal{F} := \mathcal{F}(\mathcal{T}) := \{(V,E,F,P) | V \in E \subset F \subset P\}$ consisting of all chains of incident vertices, edges, faces and tiles associated with the tiling $\mathcal{T}$. The set $\mathcal{F}$ can be geometrically interpreted in terms of a *chamber system* (or *formal barycentric subdivision*) $\mathcal{C} := \mathcal{C}_{\mathcal{T}}$, which can be obtained in the following way [see Dress (1984), Dress (1987) or Dress & Huson (1987) for details]. For each vertex, edge, face or tile associated with $\mathcal{T}$, choose an interior point, called a 0, 1, 2 or 3 *center*, respectively. Now, every flag $(V,E,F,P) \in \mathcal{F}$ defines a topological simplex $C := C(V,E,F,P) \in \mathcal{C}$ (called a *chamber*); its vertices being given by the 0, 1, 2 and 3 centers of $V$, $E$, $F$ and $P$, respectively. See Fig. 2.

Note also that, for every $i$ center in a chamber $C \in \mathcal{C}$, there is precisely one face of $C$ that is opposite to this center, which will be called the *i face* of $C$. Furthermore, for every chamber $C \in \mathcal{C}$ and any $i$ in $\{0,1,2,3\}$, there exists exactly one chamber $C' \in \mathcal{C}$ (with $C' \neq C$) such that $C$ and $C'$ have the same

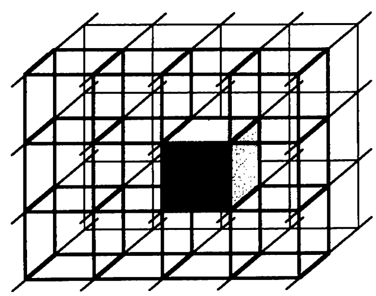


Fig. 1. The most well known face-transitive tiling of Euclidean space is the tiling by cubes.

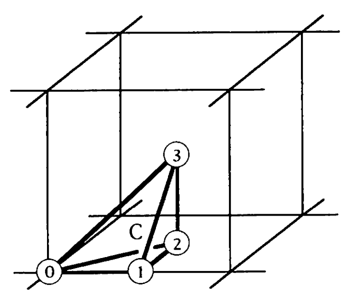



Fig. 2. A chamber $C$ of the cube tiling. The labeled circles indicate the 0, 1, 2 and 3 centers associated with $C$. Each cube consists of 48 such chambers.

$i$ face. We say that $C$ and $C'$ are *i neighbors* and we define a permutation $s_i: \mathcal{C} \to \mathcal{C}$ that maps every $C \in \mathcal{C}$ onto its $i$ neighbor $C' =: s_i(C)$ (see Fig. 3) [So $C' = s_i(C)$ implies $C = s_i(C')$.] In terms of flags, this means that for any flag $(V,E,F,P) \in \mathcal{F}$ and any $i$ in $\{0,1,2,3\}$ there exists precisely one flag $(V',E',F',P') = s_i(V,E,F,P)$ in $\mathcal{F}$ that differs from $(V,E,F,P)$ precisely in its $i$-dimensional constituent.

The natural way to let the symmetry group $\Gamma$ operate on the flag structure of the tiling and, thus, also on its chamber system, is to define

$$\gamma(V,E,F,P) := (\gamma V,\ \gamma E,\ \gamma F,\ \gamma P)$$

for any isometry $\gamma \in \Gamma$ and any flag $(V,E,F,P) \in \mathcal{F}$. Hence, we can call two chambers $C, C' \in \mathcal{C}$ *equivalent* whenever there exists a symmetry $\gamma \in \Gamma$ with $\gamma C = C'$. As we assume that $\mathcal{T}$ is periodic, it follows that the set of all equivalence classes or $\Gamma$ orbits

$$\Gamma C\{\gamma C | \gamma \in \Gamma\}\ (C \in \mathcal{C})$$

in $\mathcal{C}$ is finite. This set is called the *Delaney set* associated with $(\mathcal{T},\Gamma)$ and is denoted $\mathcal{D} := \mathcal{D}_{(\mathcal{T}\Gamma)} = \Gamma/\mathcal{C}$. In §2, we show that face-transitivity alone implies the finiteness of $\mathcal{D}$, even if periodicity of the tiling is not assumed.

Consider $\mathcal{C}$. Any symmetry $\gamma \in \Gamma$ necessarily maps $i$ neighbors onto $i$ neighbors, so the operation of $s_i$ commutes with $\Gamma$ on $\mathcal{C}$, for any $i \in \{0,1,2,3\}$. Thus, we can introduce the concept of $i$ adjacency of whole $\Gamma$ orbits. The permutation $s_i: \mathcal{D} \to \mathcal{D}$ maps every $D \in \mathcal{D}$ onto its $i$ neighbor $D' =: s_i(D)$, with $i$ in $\{0,1,2,3\}$.

The Delaney set $\mathcal{D}$ defines a finite connected four-colored graph with vertex set $\mathcal{D}$ and set of colored edges

$$\mathcal{E} := \{(\{D,D'\}, i)\ |D,D' \in \mathcal{D} \quad \text{and} \quad [s_i(D) = D'\},$$

where the component $i \in \{0,1,2,3\}$ of an edge $(\{D,D'\},i)$ is called its *color*. This *Delaney graph* $(\mathcal{D}.\mathcal{E})$ makes up the first part of the Delaney symbol corresponding to $(\mathcal{T}.\Gamma)$.

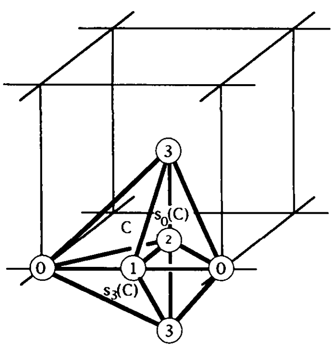



Fig. 3. The chamber $C$, depicted together with its 0 neighbor $s_0(C)$ and its three-neighbor $s_3(C)$.

The Delaney graph on its own does not hold enough information to describe a tiling completely and uniquely (up to homeomerism). We need to introduce the following functions defined on the vertices of the Delaney graph: for $0 \le i \le j \le 3$, let $m_{ij}: \mathcal{D} \to \mathbb{N}$ be defined as

$$m_{ij}(D) := \min\{m \in \mathbb{N} | (s_i s_j)^m\ (C) = C \text{ for any } C \in D\}.$$

These functions all have simple geometric interpretations (see Fig. 4).

We now come to the formal definition of a Delaney symbol. A (*three-dimensional*) *Delaney symbol* is a system $(\mathcal{D};m)$ consisting of a finite connected Delaney graph $(\mathcal{D}, \mathcal{E})$ and functions $m_{ij}: \mathcal{D} \to \mathbb{N}$, with $0 \le i < j \le 3$, such that, for every $D \in \mathcal{D}$ and $i, j$ as above, the following conditions hold true (see Figs. 4 and 5):

($DS1$) $m_{ij}(D) = m_{ij}[s_i(D)] = m_{ij}[s_j(D)]$;
($DS2$) $(s_i s_j)^{m_{ij}(D)}\ (D) = (s_j s_i)^{m_{ij}(D)}\ (D) = D$;
($DS3$) $m_{02}(D) = m_{03}(D) = m_{13}(D) = 2$;
($DS4$) $m_{01}(D) \ge 3$, $m_{12}(D) \ge 3$ and $m_{23}(D) \ge 3$.

Two Delaney symbols $(\mathcal{D};m)$ and $(\mathcal{D}';m')$ are called *isomorphic* if and only if $\#\mathcal{D} = \#\mathcal{D}'$ (where $\#\mathcal{D}$ denotes the cardinality of the set $\mathcal{D}$) and there exists a map $\pi: \mathcal{D} \to \mathcal{D}'$ with $s_k[\pi(D)] = \pi[s_k(D)]$ and $m^{i}_{ij}\ [\pi(D)] = m_{ij}(D)$ for all $D \in \mathcal{D}$, $0 \le k \le 3$ and $0 \le i < j \le 3$. The two symbols are called *homomorphic* if they fulfill the latter condition and $\#\mathcal{D} \ge \#\mathcal{D}'$. It is possible to prove the following result (see Dress, 1984, 1987):

*Lemma* 1.1. Two three-dimensional tilings $(\mathcal{T},\Gamma)$ and $(\mathcal{T}',\Gamma')$ are equivariantly equivalent if and only if the corresponding Delaney symbols $(\mathcal{D};m)$ and $(\mathcal{D}';m')$ are isomorphic. Similarly, the tilings are

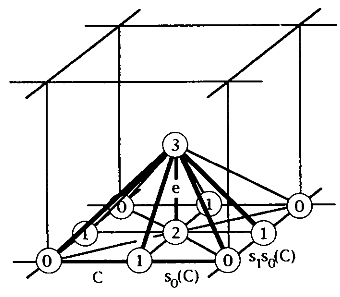



Fig. 4. Here we depict all eight chambers lying on one side of a face of a tile of the cube tiling. Note that, starting from any of the depicted chambers, say $C$, in this example, we need to apply $s_1 s_0$ precisely four times to arrive back at $C$ for the first time. This is exactly how $m_{01}$ is defined and, obviously, this number counts the number of edges of the face that is incident to the chambers $C$, $s_0(C)$, $s_1[s_0(C)]$, ..., ... . A similar discussion shows that $m_{12}$ counts the number of edges of a given tile that are incident to a given vertex and that $m_{23}$ counts the number of tiles that surround a given edge. It is also not difficult to show that the functions $m_{02}$, $m_{03}$ and $m_{13}$ are always constant and equal to two.

topologically equivalent if and only if the two symbols are homomorphic to each other or to a third symbol.

We call a Delaney symbol $(\mathcal{D};m)$ *minimal* if no homomorphic Delaney symbol exists whose Delaney set has *smaller* cardinality. Note that every topological family of tilings gives rise to precisely one (up to isomorphism unique) minimal Delaney symbol. This Delaney symbol corresponds to the so-called *maximal* tiling $[\mathcal{T},\Gamma = \text{Aut}(\mathcal{T})]$ in the topological family. This is the tiling for which the group $\Gamma$ is maximal, *i.e.* isomorphic to the full automorphism group Aut $(\mathcal{T})$ of the incidence structure of $\mathcal{T}$; that is, the group of all permutations of $\mathcal{C}_{\mathcal{T}}$ that commute with $s_0$, $s_1$, $s_2$ and $s_3$.

Let $(\mathcal{D};m)$ be the Delaney symbol of some equivariant three-dimensional tiling $(\mathcal{T},\Gamma)$. For any subset of colors $I \subset \{0,1,2,3\}$, define the *I subgraph* $(\mathcal{D},\mathcal{E}_I)$ of $(\mathcal{D},\mathcal{E})$ as the graph that one obtains by deleting all edges whose colors are not contained in $I$. Call the connected components of $(\mathcal{D},\mathcal{E}_I)$ the *I components* of $(\mathcal{D},\mathcal{E})$. (These definitions also apply to any chamber system $\mathcal{C}$.) The system consisting of an $I$ component $\mathcal{D}'$ and all functions $m_{ij}$, with $i<j$ and $i,j \in I$, restricted to $\mathcal{D}'$, is called an *I subsymbol*, where $I \subset \{0,1,2,3\}$. There is a simple relationship between certain subsymbols of $(\mathcal{D};m)$ and the different constituents of the tiling $(\mathcal{T},\Gamma)$:

*Lemma* 1.2. Let $(\mathcal{T},\Gamma)$ be an equivariant three-dimensional tiling and let $(\mathcal{D};m)$ be the corresponding Delaney symbol. There exists a one-to-one correspondence between the $\Gamma$-equivalence classes of vertices, edges, faces and tiles of $(\mathcal{T},\Gamma)$, on the one hand, and the $\{1,2,3\}$, $\{0,2,3\}$, $\{0,1,3\}$ and $\{0,1,2\}$ subsymbols of $(\mathcal{D};m)$ on the other.

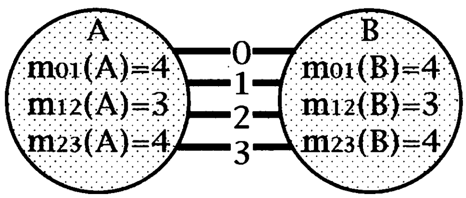



Fig. 5. Here, we depict the Delaney symbol of the tiling $(\mathcal{T},\Gamma)$ indicated in Fig. 1. We are assuming that the syummetry group $\Gamma$ consists *only* of all orientation-preserving symmetries of $\mathcal{T}$, *i.e.* that $\Gamma$ is of type 207. $P432$. In this case, the tiling $(\mathcal{T},\Gamma)$ gives rise to precisely two $\Gamma$-equivalence classes $\mathcal{A}$ and $\mathcal{B}$ of chambers. Hence, the Delaney graph of the tiling has precisely two vertices, which we depict as two circles labeled $A$ and $B$. Note that any chamber of class $\mathcal{A}$ is surrounded by chambers of class $\mathcal{B}$ and *vice versa.* In other words, we have $s_i(A) = B$ for all $i \in \{0,1,2,3\}$. The operation of $s_i$ is depicted as a line labeled $i$. Both vertices of the Delaney graph are also labeled with the values of $m_{01}$, $m_{12}$ and $m_{23}$. In both cases, the values are four (each face has four edges), three (every vertex is incident to precisely three edges of a given tile) and four (every edge is incident to precisely four tiles), respectively. This Delaney symbol is listed as no. 8 in Table 1.

*Proof.* (Sketch.) Let $(\mathcal{T},\Gamma)$ be an equivariant three-dimensional tiling with Delaney symbol $(\mathcal{D};m)$. Choose $\{i,j,k,l\} = \{0,1,2,3\}$. Let $R$ be an $l$-dimensional constituent of $\mathcal{T}$, with $l$-center $z$. Let $\mathcal{C}_z$ denote the set of all chambers with $l$-center $z$. Note that $\mathcal{C}_z$ must be an $\{i,j,k\}$ component in $\mathcal{C}$. Hence, the $\Gamma$-equivalence class $\mathcal{D}_z := \{\Gamma C | C \in \mathcal{C}_z\}$ gives rise to an $\{i,j,k\}$ subsymbol of $(\mathcal{D};m)$, which corresponds precisely to the $\Gamma$-equivalence class of $R$ in the tiling. □

The following result is crucial for all further computations:

*Lemma* 1.3. Let $(\mathcal{T},\Gamma)$ be an equivariant three-dimensional tiling with Delaney symbol $(\mathcal{D};m)$. Choose $i,j,k,l$ with $\{i,j,k,l\} = \{0,1,2,3\}$ and $i<j<k$. Every $\{i,j,k\}$ subsymbol $(\mathcal{D}';m')$ of $(\mathcal{D};m)$ must have strictly positive curvature

$$\mathcal{K}_{(\mathcal{D}';m')} := \sum_{D \in \mathcal{D}'} \{[1/m_{ij}(D)] + [1/m_{ik}(D)] + [1/m_{jk}(D)] - 1\}.$$

*Proof.* (Sketch.) Let $(\mathcal{T},\Gamma)$ be an equivariant three-dimensional tiling with Delaney symbol $(\mathcal{D};m)$. Consider any $l$-center $z$ in the corresponding chamber system $\mathcal{C}$. Let $\mathcal{C}_z$ denote the set of all chambers incident to $z$ and let $\Gamma_z < \Gamma$ denote the stabilizer group of $z$. One can view the system $(\mathcal{C}_z,\Gamma_z)$ as a two-dimensional equivariant tiling, by considering the $l$ faces of the chambers in $\mathcal{C}_z$ to be two-dimensional tiles. This tiling must be spherical. This implies positive curvature by the Euler theorem, as observed by Dress (1987, p. 210). □

## 2. The finiteness theorem

In this section, we prove that the number of classes of face-transitive (indeed, more generally, face-$k$-transitive) three-dimensional tilings is finite.

*Lemma* 2.1. Let $(\mathcal{D},\mathcal{E})$ be a finite connected Delaney graph. There are only finitely many possible choices for the functions $m_{ij}: \mathcal{D} \to \mathbb{N}$ $(0 \le i < j \le 3)$ such that $(\mathcal{D};m)$ is a Delaney symbol corresponding to an equivariant three-dimensional tiling $(\mathcal{T},\Gamma)$ of $\mathbb{E}^3$.

*Proof.* Let $(\mathcal{D};m)$ be a Delaney symbol. Define $r_{ij}(D) := \min\{r \in \mathbb{N} | (s_is_j)^r(D) = D\}$ for all $D \in \mathcal{D}$ and $0 \le i < j \le 3$. By definition, $r_{ij}(D) \le \#\mathcal{D}$ and $r_{ij}(D)$ divides $m_{ij}(D)$ so we can define the *branching-number* function $v_{ij}(D) := m_{ij}(D)/r_{ij}(D)$ for all $D \in \mathcal{D}$ and $0 \le i < j \le 3$. From $(DS1)$ and $(DS2)$, it follows that the functions $m_{ij}$ and $r_{ij}$ are constant on the $\{i,j\}$ components and, hence, $v_{ij}$ is also. It is sufficient to prove that the branching-number function can assume only finitely many values. It is not difficult to see that, for any $\{i,j\}$ component $\mathcal{C}$, the value of $v_{ij}(\mathcal{C}) := v_{ij}(D)$, with $D \in \mathscr{C}$, is always equal to the order of some rotation appearing in the group $\Gamma$. Choose $\{i,j,k,l\} = \{0,1,2,3\}$. Consider some specific edge $e$ of a chamber $C \in \mathscr{C}$ that joins, say, a $k$ center and an $l$ center. Now consider the set of all chambers $\mathscr{C}_e$ that are incident to the edge $e$. (Consider, for example, Fig. 4, with $k=2$ and $l=3$.) With $C \in D \in \mathscr{D}$, there are precisely $2m_{ij}(D)$ chambers in $\mathscr{C}_e$, partitioned into $r_{ij}(D)$ or $2r_{ij}(D)$ different $\Gamma$-equivalence classes, depending on whether the stabilizer group $\Gamma_e < \Gamma$ is a dihedral group or a cyclic group. In both cases, the rotational order of the stabilizer group $\Gamma_e$ of the edge $e$ must equal $v_{ij}(D)$. So, the lemma is proved because the well known *crystallographic restriction* states that the only rotational orders that can appear in a crystallographic space group are 1, 2, 3, 4 and 6. □

The proof of the following result does not make use of properties of Euclidean space, so it is also true for Delaney symbols encoding tilings of, say, the hyperbolic space $\mathbb{H}^3$. It uses a variant of an argument already used in a similar context by Dress & Scharlau (1984) (see also Dress & Huson, 1991).

*Lemma* 2.2. Let $(\mathscr{D};m)$ be the Delaney symbol of some equivariant three-dimensional tiling. If the number of $\{0,1,3\}$ components in $\mathscr{D}$ is $k$, then

$$\#\mathscr{D} < 24k.$$

*Proof.* Assume the number of $\{0,1,3\}$ components is less than or equal to $k$. From ($DS$3), it follows that $s_3$ commutes with $s_0$ and $s_1$. Hence, $s_3$ induces a permutation of order at most two on the set of $\{0,1\}$ components in $\mathscr{D}$ and, therefore, the number $r$ of $\{0,1\}$ components in $\mathscr{D}$ cannot exceed $2k$. From lemma 1.3, it follows, with $m_{02}(D)=2$ for all $D \in \mathscr{D}$, that

$$\sum_{D\in\mathscr{C}} \{[1/m_{01}(D)] + [1/m_{12}(D)] - (1/2)\} > 0$$

for all $\{0,1,2\}$ components $\mathscr{C} \subset \mathscr{D}$. Therefore,

$$\sum_{D\in\mathscr{D}} \{[1/m_{01}(D)] + [1/m_{12}(D)] - (1/2)\} > 0.$$

The latter equation, together with $m_{12}(D) \geq 3$ for all $D \in \mathscr{D}$, leads to

$$0 < \sum_{D\in\mathscr{D}} [1/m_{01}(D)] + \sum_{D\in\mathscr{D}} [1/m_{12}(D)] - \sum_{D\in\mathscr{D}} 1/2$$
$$\leq 2r + (1/3)\#\mathscr{D} - (1/2)\#\mathscr{D} = 2r - (1/6)\#\mathscr{D}$$

and, thus,

$$\#\mathscr{D}/6 < 2r \text{ or } \#\mathscr{D} < 12r \leq 24k$$

if we simply observe [see Dress & Scharlau (1984) or Dress & Huson (1987) for details] that

$$\sum_{D\in\mathscr{D}} [1/m_{01}(D)] = \sum_{\mathscr{C}} \sum_{D\in\mathscr{C}} [1/m_{01}(D)]$$
$$= \sum_{\mathscr{C}} [\#\mathscr{C}/m_{01}(D\in\mathscr{C})] \leq \sum_{\mathscr{C}} 2 = 2r,$$

where we sum over all $\{0,1\}$ components $\mathscr{C}$. □

The result, together with lemma 2.1, implies

*Theorem* 2.3. For any $k \in \mathbb{N}$, the number of equivariant types of face-$k$-transitive three-dimensional tilings is finite.

## 3. Computation of feasible three-dimensional Delaney symbols

In this section, we first discuss how to obtain a list of *candidate* Delaney symbols that contains, by construction, all Delaney symbols of equivariant face-transitive three-dimensional tilings (up to isomorphism). This list is generated by first computing the possible $\{0,1,2\}$ subgraphs and functions $m_{01}$ and $m_{12}$ and then adding the edges of color 3 and defining the function $m_{23}$. We then go on to indicate how to single out precisely those Delaney symbols that do indeed correspond to face-transitive three-dimensional tilings [For more traditional $d$-dimensional algorithms, see Molnár (1992) and Molnár & Prok (1993).]

Let $(\mathscr{D};m)$ be a Delaney symbol corresponding to some face-transitive tiling $(\mathscr{T},\Gamma)$. Lemma 1.2 implies that $\mathscr{D}$ consists of exactly one $\{0,1,3\}$ component. From this, together with the fact that $s_3$ commutes with both $s_0$ and $s_1$, it follows that the number $N_{01}$ of $\{0,1\}$ components in $\mathscr{D}$ is at most two and that $s_3$ defines a $\{0,1\}$ *isomorphism* on the one or between the two component(s). Note that the number $N_{012}$ of $\{0,1,2\}$ components cannot exceed $N_{01}$, trivially. Hence, there are three possible cases:

*One.* $N_{01}=1$ and $N_{012}=1$. In this case, the $\{0,1,2\}$ subsymbol of $(\mathscr{D};m)$ is the two-dimensional Delaney symbol of an equivariant topological polyhedron, whose faces are all equivalent with respect to the stabilizer group of the polyhedron. Following Dress (1987), all such two-dimensional Delaney symbols can easily be generated by computer (see Huson, 1993) (*cf.* also Grünbaum & Shephard, 1981*a*).

*Two.* $N_{01}=2$ and $N_{012}=2$. In this case, both $\{0,1,2\}$ subsymbols are two-dimensional Delaney symbols of topological polyhedra as described in (one),

*Double.* $N_{01}=2$ and $N_{012}=1$. In this case, the $\{0,1,2\}$ subsymbol is the two-dimensional Delaney symbol of an equivariant topological polyhedron that possesses precisely two equivalence classes of faces (with respect to the stabilizer group of the polyhedron), all faces having the same number of edges. All such two-dimensional Delaney symbols can also be generated easily by computer (see Dress, 1987; Huson, 1993).

Now, in all three cases, proceed as follows. Assume that, as described above, we are given a three-colored graph $(\mathscr{D},\mathscr{E})$, with $\#\mathscr{D} < 24$, consisting of one or two $\{0,1,2\}$ components, together with appropriate functions $m_{01}$, $m_{12}$, $m_{02}$. Successively generate and consider all possible definitions of edges of color 3 on $\mathscr{D}$. For each 'promising' definition, define $m_{23}: \mathscr{D} \to \mathbb{N}$ in all feasible ways, making sure that the resulting four-colored graph is connected and that properties (*DS*1)–(*DS*4) hold.

The fact that only branching numbers 1, 2, 3, 4 and 6 can occur ensures that, for any given definition of the edges of color 3, only a small number of possible choices of $m_{23}$ have to be considered. Furthermore, one must only consider such definitions of $m_{23}$ that lead to positive *curvature* of each {1,2,3} component, reflecting the fact that the stabilizer groups of the vertices of the corresponding tiling are (finite) crystallographic point groups.

Using computer programs *ONE*, *TWO* and *DOUBLE*, based on the above remarks, we obtain the following result:

*Lemma* 3.1. In each of the cases One, Two and Double, there exist at most 171, 183 and 185 Delaney symbols, respectively, that might possibly correspond to Euclidean face-transitive three-dimensional tilings.

Given such a list of candidates, one can proceed as follows. As indicated in the proof of lemma 2.1, the branching numbers of a Delaney symbol always correspond to rotational orders in the symmetry group of the corresponding tiling. Indeed, each $\{i,j,k\}$ subsymbol determines, up to equivalence, the induced stabilizer group of the corresponding vertex, edge, face or tile [see Dress & Scharlau (1984) or Molnár (1991) for details]. Using a computer program *STAB-GROUPS* based on this, for each of the candidate Delaney symbols, we can compute all equivalence classes of stabilizer groups in the symmetry group of the corresponding tiling. Note that any nontrivial stabilizer group $\Gamma_p$ leaves either exactly one point, or one line, or one plane in $\mathbb{E}^3$ point-wise fixed. We need to distinguish between the first type of group, which we will refer to as a *strict* point group, and the latter two.

Let $(\mathscr{D};m)$ be one of the candidate Delaney symbols. A necessary condition for the existence of a corresponding equivariant tiling is that there must exist some crystallographic space group (see Hahn, 1983) containing precisely the same combination of equivalence classes of *strict* stabilizer point groups as computed for $(\mathscr{D};m)$. Furthermore, the other two types of stabilizer groups must be compatible with the rotational axes and reflectional planes of the space group. In general, there may exist more than one feasible group.

Let us call an equivariant tiling $(\mathscr{T},\Gamma)$ *orientable* if its symmetry group $\Gamma$ onsists of orientation-preserving isometries only, *i.e.* of isometries whose linear components have determinant 1. We call a Delaney symbol $(\mathscr{D};m)$ *orientable* if its Delaney graph is bipartite in the usual sense, *i.e.* if there exists an *orientation map* $\omega$: $\mathscr{D} \to \{-1,+1\}$ such that $\omega[s_i(D)] = -\omega(D)$ for all $D \in \mathscr{D}$ and $i \in \{0,1,2,3\}$.

Assume that $(\mathscr{D};m)$ encodes an equivariant tiling $(\mathscr{T},\Gamma)$. It can be shown that $(\mathscr{D};m)$ is orientable if and only if $(\mathscr{T},\Gamma)$ is orientable. Let $(\mathscr{T},\Gamma)$ be a nonorientable equivariant tiling with Delaney symbol $(\mathscr{D};m)$. The following construction yields the Delaney symbol $(\mathscr{D}';m')$, called the *oriented covering* of $(D;m)$, corresponding to the tiling $(\mathscr{T},\Gamma')$ that one obtains from $(\mathscr{T},\Gamma)$ by removing all orientation-reversing isometries in $\Gamma$. Set $\mathscr{D}' := D \times \{-1,+1\}$ and define $s_i(D,\varepsilon) := (s_iD, -\varepsilon)$ for all $D \in \mathscr{D}$, $\varepsilon \in \{-1,+1\}$ and $i \in \{0,1,2,3\}$. Set $m'_{ij}[(D,\varepsilon)] := m_{ij}(D)$ for all $D \in \mathscr{D}$, $\varepsilon \in \{-1,+1\}$ and $0 \le i < j \le 3$.

Let $(\mathscr{D};m)$ be a Delaney symbol and assume that $(\mathscr{D};m)$ is not orientable. If $(\mathscr{D};m)$ encodes an equivariant tiling of Euclidean space, then so, too, must its oriented covering $(\mathscr{D}';m')$. Hence, in the attempt to exclude nonrealizable Delaney symbols, we can focus our investigations on orientable Delaney symbols.

For any given *orientable* three-dimensional Delaney symbol $(\mathscr{D};m)$ of an *orientable* equivariant tiling, one can compute the so-called *symmetry skeleton* of the tiling. This is the graph whose labeled vertices correspond to the nonequivalent strict stabilizer point groups and whose labeled edges correspond to the nonequivalent sections of rotational axes. An edge is incident to a vertex in the graph if and only if the corresponding rotational axis is incident to the point stabilized by the group corresponding to the vertex (see Figs. 6 and 7). It can be shown that any two (orientable) tilings with symmetry groups of the same crystallographic type give rise to symmetry skeletons that are *isomorphic* as labeled graphs (but not *vice versa*). Hence, given an orientable three-dimensional Delaney symbol $(\mathscr{D};m)$, we need to determine whether the corresponding symmetry skeleton is compatible with one of the 219 isomorphism types of crystallographic space groups. The symmetry skeleton associated with a given type

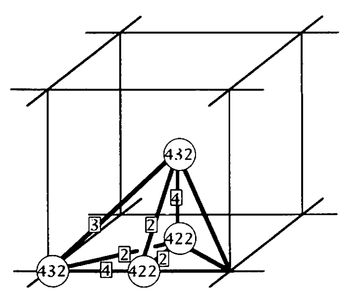



Fig. 6. A fundamental region of the tiling $(\mathscr{T},\Gamma)$ dicussed in Figs. 1–5 is indicated by heavy lines. The tiling has two $\Gamma$ classes of strict point stabilizer groups of type 432 and two of type 422, indicated by labeled circles. Furthermore, the tiling has three $\Gamma$ classes of twofold rotational axes, two of fourfold axes and one of threefold axes, indicated by labeled lines.

of crystallographic space group can either be determined from the information given by Hahn (1983) or, more easily, from the Delaney symbol of a tiling known to exist for the given space-group type. Furthermore, Dunbar (1988) lists the (embedded) symmetry skeletons of all crystallographic groups whose 'orbit spaces' are homeomorphic to the three-dimensional sphere or projective space.

Determination of which of the 219 crystallographic space groups contain a given combination of stabilizer groups was initially quite a laborious process worked out by Molnár, greatly aided by a systematic collection based on the work of Henry & Lonsdale (1979). Recently, Huson has developed a computer program *SPACE-GROUP PREDICTOR*, which automatically performs this task and also computes the symmetry skeleton corresponding to the given Delaney symbol.

Initially, Molnár carefully inspected all candidate Delaney symbols by hand and concluded that, of these, precisely 88 yield realizable tilings. Later, the data were re-examined in the following way. First, using the program *SPACE-GROUP PREDICTOR*, we determined that, of those symbols, precisely 169 imply stabilizer groups compatible with one or more of the crystallographic space groups. Second, by computer, we determined that precisely 165 of them are minimal. The number of Delaney symbols that are both minimal and also pass the *SPACE-GROUP PREDICTOR* test is exactly 12. After careful inspection of these 12 Delaney symbols, we conclude that precisely seven are realizable. The other five each give rise to a non-Euclidean tiling and, hence, cannot be the symbol of some tiling of Euclidean space (see Tables 1 to 4).

Although we are certain that it is sufficient to consider only the *minimal* Delaney symbols in this investigation, we cannot prove this fact at present. Therefore, the initial inspection of all candidate Delaney symbols by Molnár played an important role.

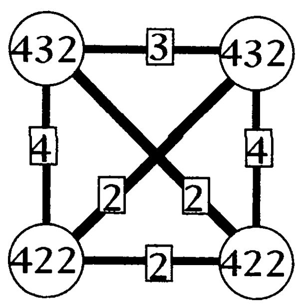



Fig. 7. The symmetry skeleton associated with the tiling $(\mathcal{T},\Gamma)$ and, thus, with the crystallographic group type 207.$P$432. The vertices represent the different $\Gamma$-equivalence classes of the point stabilizers of the group and the edges represent the $\Gamma$-equivalence classes of sections of rotational axes.

From these considerations, we finally obtain the main result:

*Theorem* 3.2. There exist precisely 88 classes of face-transitive three-dimensional tilings. These fall into seven topological families. Of the 88 types, precisely 12 can only be realized using 'marked' tiles, five others possess only convex realizations, a further six have both convex and nonconvex realizations whereas the other 65 can only be realized using nonconvex tiles.

This analysis is closely connected to the classification of *orbifolds*, *i.e.* of topological spaces that 'locally look like' the orbit space of a symmetry group (see Thurston, 1980; Scott, 1983). The Delaney symbol of an equivariant tiling is, in fact, a sort of triangulation, induced by the tiling, of the orbifold associated with the symmetry group of the tiling. A deeper understanding of this connection between orbifolds, Delaney symbols and tilings will be the aim of future investigations. We hope that this will prove useful for a detailed – and in particular computationally oriented – study of both concepts.

## 4. Results and examples

There exist precisely seven topological families of face-transitive three-dimensional tilings. The types of tiles involved are: the cube, the tetrahedron, the rhombic dodecahedron, the octahedron, the special rhombohedron, the covered rhombohedron and, for the seventh type of tiling, the octahedron and the tetrahedron. These are all well known tiles except, perhaps, the special rhombohedron (see Fig. 8) and the covered rhombohedron (see Fig. 9 and example 3 below).

In Table 1, we list the 88 Delaney symbols that correspond to the classes of equivariant face-transitive three-dimensional tilings. Then, in Table 2, for each Delaney symbol we indicate the topological family of the tiling, the stabilizer groups and the crystallographic space group. Furthermore, we indicate whether the Delaney symbol gives rise to a 'marked', convex or nonconvex type of three-dimensional tiling.

In Tables 3 and 4, we list the five Delaney symbols whose symmetry skeletons are compatible with some crystallographic group, but which do not give rise to face-transitive tilings of Euclidean space.

Finally, in Fig. 11, we indicate which Delaney symbol is homomorphic to which, *i.e.* which tiling can be derived from which other tiling by symmetry breaking.

*Example* 1. Let $(\mathcal{D};m)$ be the Delaney symbol no. 1 in Table 1. As we will now show, it encodes the familiar cube tiling $(\mathcal{T},\Gamma)$. From $\mathcal{D} = \{D\}$, $m_{01}(D) =$

Table 1. *The 88 nonisomorphic Delaney symbols corresponding to face-transitive three-dimensional tilings*

Delaney symbols nos. 1–40, 41–69 and 70–88 correspond to the cases One, Two and Double, respectively. Other than that, there is no special ordering. To save space, the symbols have been encoded. Each line defines one Delaney symbol. Consider, for example, line 28:

$$28 \quad D = 6{:}246,635,265,265m = 3,44,433.$$

The first number following '$D$ =' indicates the number of vertices of the Delaney graph. In this case, we have $\mathcal{D} = \{1,2,3,\ldots,6\}$. The subsequent group of numbers, 246, defines the 0 edges in ascending order: $s_0(1) = 2$, $s_0(3) = 4$ and $s_0(5) = 6$. The second group of numbers, 635, defines the 1 edges: $s_1(1) = 6$, $s_1(2) = 3$ and $s_1(4) = 5$. Similarly, the other two groups of numbers define the 2 and 3 edges. Following '$m$ =', we list the values of the functions $m_{01}$, $m_{12}$ and then $m_{23}$, on each $\{i,j\}$ component, in ascending order of the components. In the example, $m_{01} = 3$ on the only $\{0,1\}$ component, $m_{12} = 4$ on both $\{1,2\}$ components in $\mathcal{D}$ and $m_{23} = 4$ on the first and 3 on the other two $\{2,3\}$ components. In the latter case, $m_{23}(1) = m_{23}(2) = 4$ and $m_{23}(3) = \ldots = m_{23}(6) = 3$.

| No. | Delaney symbol |
|---|---|
| 1 | D = 1 : 1, 1, 1, 1 m = 4, 3, 4 |
| 2 | D = 2 : 1 2, 2, 2, 1 2 m = 4, 3, 4 |
| 3 | D = 2 : 1 2, 2, 2, 2 m = 4, 3, 4 |
| 4 | D = 2 : 2, 1 2, 1 2, 1 2 m = 4, 3 4, 3 3 |
| 5 | D = 2 : 2, 1 2, 1 2, 1 2 m = 4, 3 3, 4 4 |
| 6 | D = 2 : 2, 1 2, 1 2, 2 m = 4, 3 3, 4 |
| 7 | D = 2 : 2, 2, 2, 1 2 m = 4, 3, 4 |
| 8 | D = 2 : 2, 2, 2, 2 m = 4, 3, 4 |
| 9 | D = 3 : 1 3, 2 3, 1 2 3, 1 2 3 m = 3, 4 4, 4 3 3 |
| 10 | D = 3 : 1 3, 2 3, 1 3, 1 2 3 m = 3, 3, 4 6 |
| 11 | D = 4 : 2 4, 1 3 4, 1 2 3 4, 1 2 3 4 m = 4, 3 4 3, 3 3 3 3 |
| 12 | D = 4 : 2 4, 1 3 4, 1 2 4, 1 2 3 4 m = 4, 3 3, 4 4 4 |
| 13 | D = 4 : 2 4, 1 3 4, 1 2 4, 1 2 3 4 m = 4, 3 3, 3 3 6 |
| 14 | D = 4 : 2 4, 1 3 4, 1 2 4, 4 3 m = 4, 3 3, 4 |
| 15 | D = 4 : 2 4, 1 3 4, 4 3, 4 3 m = 4, 4 3, 3 3 |
| 16 | D = 4 : 2 4, 4 3, 4 3, 1 2 3 4 m = 4, 3 3, 4 4 |
| 17 | D = 4 : 2 4, 4 3, 4 3, 2 4 m = 4, 3 3, 4 |
| 18 | D = 4 : 2 4, 4 3, 4 3, 3 4 m = 4, 3 3, 4 |
| 19 | D = 4 : 2 4, 4 3, 4 3, 4 3 m = 4, 3 4, 3 3 |
| 20 | D = 4 : 2 4, 4 3, 4 3, 4 3 m = 4, 3 3, 4 4 |
| 21 | D = 6 : 2 4 6, 6 3 5, 1 2 3 4 5 6, 1 2 3 4 5 6<br>m = 3, 4 4 4, 4 4 3 3 3 3 |
| 22 | D = 6 : 2 4 6, 6 3 5, 1 2 3 4 6, 1 2 3 4 5 6<br>m = 3, 4 4, 3 3 3 3 4 |
| 23 | D = 6 : 2 4 6, 6 3 5, 1 2 6 5, 2 6 5 m = 3, 4 4, 4 3 3 |
| 24 | D = 6 : 2 4 6, 6 3 5, 2 4 6, 1 2 3 4 5 6 m = 3, 3, 6 6 4 |
| 25 | D = 6 : 2 4 6, 6 3 5, 2 4 6, 2 6 5 m = 3, 3, 4 6 |
| 26 | D = 6 : 2 4 6, 6 3 5, 2 5 6, 1 2 3 4 5 6 m = 3, 3, 4 6 6 |
| 27 | D = 6 : 2 4 6, 6 3 5, 2 5 6, 2 6 5 m = 3, 3, 4 6 |
| 28 | D = 6 : 2 4 6, 6 3 5, 2 6 5, 2 6 5 m = 3, 4 4, 4 3 3 |
| 29 | D = 8 : 2 4 6 8, 8 3 5 7, 2 4 8 7, 1 2 3 4 5 6 7 8<br>m = 4, 3 3, 4 4 4 4 |
| 30 | D = 8 : 2 4 6 8, 8 3 5 7, 2 4 8 7, 4 3 8 7 m = 4, 3 3, 6 3 3 |
| 31 | D = 8 : 2 4 6 8, 8 3 5 7, 2 4 8 7, 4 3 8 7 m = 4, 3 3, 4 4 4 |
| 32 | D = 8 : 2 4 6 8, 8 3 5 7, 2 4 8 7, 5 6 7 8 m = 4, 3 3, 4 |
| 33 | D = 8 : 2 4 6 8, 8 3 5 7, 2 4 8 7, 8 7 6 5 m = 4, 3 3, 4 |
| 34 | D = 8 : 2 4 6 8, 8 3 5 7, 3 4 8 7, 1 2 3 4 5 6 7 8<br>m = 4, 3 3, 4 4 4 4 |
| 35 | D = 8 : 2 4 6 8, 8 3 5 7, 3 4 8 7, 2 8 7 6 m = 4, 3 3, 4 |
| 36 | D = 8 : 2 4 6 8, 8 3 5 7, 3 4 8 7, 4 3 8 7 m = 4, 3 3, 6 3 3 |
| 37 | D = 8 : 2 4 6 8, 8 3 5 7, 3 4 8 7, 4 3 8 7 m = 4, 3 3, 4 4 4 |
| 38 | D = 8 : 2 4 6 8, 8 3 5 7, 3 4 8 7, 5 6 7 8 m = 4, 3 3, 4 |
| 39 | D = 8 : 2 4 6 8, 8 3 5 7, 3 4 8 7, 8 7 6 5 m = 4, 3 3, 4 |
| 40 | D = 8 : 2 4 6 8, 8 3 5 7, 4 3 8 7, 4 3 8 7 m = 4, 4 3 3, 3 3 3 3 |
| 41 | D = 2 : 1 2, 1 2, 1 2, 2 m = 4 4, 3 3, 4 |
| 42 | D = 2 : 1 2, 1 2, 1 2, 2 m = 3 3, 4 3, 4 |
| 43 | D = 4 : 1 2 3 4, 2 4, 2 4, 3 4 m = 4 4, 3 3, 4 |
| 44 | D = 4 : 2 4, 1 2 3 4, 1 2 3 4, 3 4 m = 4 4, 3 3 3 3, 4 4 |
| 45 | D = 4 : 2 4, 2 4, 1 2 4, 3 4 m = 3 3, 4 3, 4 |
| 46 | D = 4 : 2 4, 2 4, 2 4, 3 4 m = 4 4, 3 3, 4 |
| 47 | D = 4 : 2 4, 2 4, 2 4, 3 4 m = 3 3, 4 3, 4 |
| 48 | D = 6 : 1 3 4 6, 2 3 5 6, 1 2 3 4 6, 4 5 6 m = 3 3, 4 4 3, 4 4 |
| 49 | D = 6 : 1 3 4 6, 2 3 5 6, 1 3 4 6, 4 5 6 m = 3 3, 3 3, 4 6 |
| 50 | D = 8 : 2 4 6 8, 1 3 4 5 7 8, 1 2 4 5 6 8, 5 6 7 8<br>m = 4 4, 3 3 3 3, 4 4 4 |
| 51 | D = 8 : 2 4 6 8, 1 3 4 5 7 8, 1 2 4 5 6 8, 8 7 6 5<br>m = 4 4, 3 3 3 3, 4 4 |
| 52 | D = 8 : 2 4 6 8, 4 3 8 7, 4 3 8 7, 5 6 7 8 m = 4 4, 3 3 3 3, 4 4 |
| 53 | D = 12 : 2 4 6 8 10 12, 6 3 5 12 9 11, 1 2 3 4 5 6 8 10 12,<br>7 8 9 10 11 12 m = 3 3, 4 4 4 3, 4 4 4 |
| 54 | D = 12 : 2 4 6 8 10 12, 6 3 5 12 9 11, 1 2 3 4 5 6 8 11 12,<br>7 8 9 10 11 12 m = 3 3, 4 4 4 3, 4 4 4 |
| 55 | D = 12 : 2 4 6 8 10 12, 6 3 5 12 9 11, 1 2 3 4 6 8 10 12,<br>7 8 9 10 11 12 m = 3 3, 4 4 3, 4 4 4 |
| 56 | D = 12 ; 2 4 6 8 10 12, 6 3 5 12 9 11, 1 2 3 4 6 8 11 12,<br>9 10 11 12 7 8 m = 3 3, 4 4 3, 4 4 4 |
| 57 | D = 12 : 2 4 6 8 10 12, 6 3 5 12 9 11, 1 2 6 5 8 10 12,<br>7 8 9 10 11 12 m = 3 3, 4 4 3, 4 4 |
| 58 | D = 12 : 2 4 6 8 10 12, 6 3 5 12 9 11, 1 2 6 5 8 11 12,<br>7 8 9 10 11 12 m = 3 3, 4 4 3, 4 4 |
| 59 | D = 12 : 2 4 6 8 10 12, 6 3 5 12 9 11, 2 4 6 8 10 12,<br>7 8 9 10 11 12 m = 3 3, 3 3, 6 6 4 |
| 60 | D = 12 : 2 4 6 8 10 12, 6 3 5 12 9 11, 2 4 6 8 12 11,<br>7 8 9 10 11 12 m = 3 3, 3 4 4, 4 4 |
| 61 | D = 12 ; 2 4 6 8 10 12, 6 3 5 12 9 11, 2 5 6 8 11 12,<br>7 8 9 10 11 12 m = 3 3, 3 3, 4 6 6 |
| 62 | D = 12 : 2 4 6 8 10 12, 6 3 5 12 9 11, 2 5 6 8 12 11,<br>7 8 9 10 11 12 m = 3 3, 3 4 4, 4 4 |
| 63 | D = 16 : 2 4 6 8 10 12 14 16, 8 3 5 7 16 11 13 15,<br>2 4 8 7 10 12 16 15, 9 10 11 12 13 14 15 16<br>m = 4 4, 3 3 3 3, 4 4 4 4 |
| 64 | D = 16 : 2 4 6 8 10 12 14 16, 8 3 5 7 16 11 13 15,<br>2 4 8 7 10 12 16 15, 13 14 15 16 9 10 11 12<br>m = 4 4, 3 3 3 3, 4 4 |
| 65 | D = 16 ; 2 4 6 8 10 12 14 16, 8 3 5 7 16 11 13 15,<br>2 4 8 7 11 12 16 15, 9 10 11 12 13 14 15 16<br>m = 4 4, 3 3 3 3, 4 4 4 |
| 66 | D = 16 ; 2 4 6 8 10 12 14 16, 8 3 5 7 16 11 13 15,<br>2 4 8 7 11 12 16 15, 13 14 15 16 9 10 11 12<br>m = 4 4, 3 3 3 3, 4 4 |
| 67 | D = 16 : 2 4 6 8 10 12 14 16, 8 3 5 7 16 11 13 15,<br>3 4 8 7 11 12 16 15, 9 10 11 12 13 14 15 16<br>m = 4 4, 3 3 3 3, 4 4 4 4 |
| 68 | D = 16 ; 2 4 6 8 10 12 14 16, 8 3 5 7 16 11 13 15,<br>3 4 8 7 11 12 16 15, 10 9 16 15 14 13 12 11<br>m = 4 4, 3 3 3 3, 4 4 |
| 69 | D = 16 ; 2 4 6 8 10 12 14 16, 8 3 5 7 16 11 13 15,<br>3 4 8 7 11 12 16 15, 13 14 15 16 9 10 11 12<br>m = 4 4, 3 3 3 3, 4 4 |
| 70 | D = 8 : 2 4 6 8, 1 3 4 5 7 8, 1 2 5 6 7 8, 5 6 7 8<br>m = 4 4, 3 3 3.3, 4 4 |
| 71 | D = 8 : 2 4 6 8, 1 3 4 5 7 8, 1 2 5 6 7 8, 8 7 6 5<br>m = 4 4, 3 3 3 3, 4 4 4 |
| 72 | D = 6 : 1 3 4 6, 2 3 5 6, 1 5 6 4, 4 5 6 m = 3 3, 4 4, 4 3 3 |
| 73 | D = 6 ; 1 3 4 6, 2 3 5 6, 1 6 5 4, 4 5 6 m = 3 3, 3 3, 4 6 |
| 74 | D = 6 ; 1 3 4 6, 2 3 5 6, 4 5 6, 4 5 6 m = 3 3, 4 4, 4 3 3 |
| 75 | D = 12 : 2 4 6 8 10 12, 6 3 5 12 9 11, 1 2 7 8 9 10 11 12,<br>11 12 7 8 9 10 m = 3 3, 4 4 4, 4 4 3 3 3 3 |
| 76 | D = 12 : 2 4 6 8 10 12, 6 3 5 12 9 11, 2 7 8 9 10 12,<br>11 12 7 8 9 10 m = 3 3, 4 4, 4 3 3 3 3 |
| 77 | D = 12 : 2 4 6 8 10 12, 6 3 5 12 9 11, 2 7 8 11 12 10,<br>9 10 11 12 7 8 m = 3 3, 3 3, 4 6 6 |
| 78 | D = 12 : 2 4 6 8 10 12, 6 3 5 12 9 11, 2 7 8 11 12 10,<br>10 9 8 7 12 11 m = 3 3, 3 3, 6 6 4 |
| 79 | D = 12 : 2 4 6 8 10 12, 6 3 5 12 9 11, 2 7 8 11 12 10,<br>10 9 8 7 12 11 m = 3 3, 3 3, 4 6 6 |
| 80 | D = 12 ; 2 4 6 8 10 12, 6 3 5 12 9 11, 3 4 7 8 12 11,<br>8 7 12 11 10 9 m = 3 3, 5 3, 3 3 |
| 81 | D = 12 : 2 4 6 8 10 12, 6 3 5 12 9 11, 7 8 9 10 11 12,<br>7 8 9 10 11 12 m = 3 3, 4 4 4, 4 4 3 3 3 3 |
| 82 | D = 12 : 2 4 6 8 10 12, 6 3 5 12 9 11, 7 8 9 10 12 11,<br>7 8 9 10 11 12 m = 3 3, 4 4, 3 3 3 3 4 |
| 83 | D = 16 : 2 4 6 8 10 12 14 16, 8 3 5 7 16 11 13 15,<br>4 3 9 10 14 13 15 16, 12 11 10 9 16 15 14 13<br>m = 4 4, 4 3 3, 3 3 3 3 |
| 84 | D = 16 ; 2 4 6 8 10 12 14 16, 8 3 5 7 16 11 13 15,<br>4 3 9 10 15 16 14 13, 10 9 16 15 14 13 12 11<br>m = 4 4, 3 3 3 3, 4 4 |
| 85 | D = 16 : 2 4 6 8 10 12 14 16, 8 3 5 7 16 11 13 15,<br>4 3 9 10 15 16 14 13, 11 12 13 14 15 16 9 10<br>m = 4 4, 3 3 3 3, 4 4 4 4 |
| 86 | D = 16 : 2 4 6 8 10 12 14 16, 8 3 5 7 16 11 13 15,<br>4 3 9 10 15 16 14 13, 12 11 10 9 16 15 14 13<br>m = 4 4, 3 3 3 3, 4 4 |
| 87 | D = 16 : 2 4 6 8 10 12 14 16, 8 3 5 7 16 11 13 15,<br>4 3 9 10 15 16 14 13, 14 13 12 11 10 9 16 15<br>m = 4 4, 3 3 3 3, 4 4 4 4 |
| 88 | D = 16 ; 2 4 6 8 10 12 14 16, 8 3 5 7 16 11 13 15,<br>4 3 9 10 15 16 14 13, 15 16 9 10 11 12 13 14<br>m = 4 4, 3 3 3 3, 4 4 |

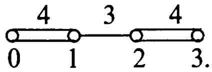
Table 2. *Space groups, stabilizer groups, topological families and types of realizations for the* 88 *face-transitive three-dimensional tilings* $(\mathcal{T},\Gamma)$ *corresponding to the Delaney symbols in Table* 1

The crystallographic space group $\Gamma$ is given in column (1). For notation, see Hahn (1983, p. 103). The stabilizer groups for the $\Gamma$ classes of the tiles (3), faces (2), edges (1) and vertices (0) of $\mathcal{T}$ are listed in column (2). For notation, see Hahn (1983, p. 776). In column (3), we note to which of the seven topological families the tiling belongs. Asterisks indicate tilings that are the maximal representatives of their topological families. Finally, in column (4), the letters '*c*' '*n*' and '*m*' indicate whether the corresponding type of tiling possesses convex, nonconvex or only 'marked' realizations.

| No. | (1) | (2) | (3)† | (4) |
|---|---|---|---|---|
| 1 | 221.$Pm\bar{3}m$ | (3) $m\bar{3}m$ (2) $4/mmm$ (1) $4/mmm$ (0) $m\bar{3}m$ | cube* | *c* |
| 2 | 200.$Pm\bar{3}$ | (3) $m\bar{3}$ (2) $mmm$ (1) $mmm$ (0) $m\bar{3}$ | cube | *m* |
| 3 | 226.$Fm\bar{3}c$ | (3) $m\bar{3}$ (2) $\bar{4}2m$ (1) $4/m$ (0) 432 | cube | *n* |
| 4 | 225.$Fm\bar{3}m$ | (3) $m\bar{3}m$ (2) $mmm$ (1) $3m$ ($0_1$) $\bar{4}3m$ ($0_2$) $m\bar{3}m$ | rhomb. dodec.* | *c* |
| 5 | 225.$Fm\bar{3}m$ | (3) $\bar{4}3m$ (2) $mmm$ (1) $4mm$ ($0_1$) $m\bar{3}m$ ($0_2$) $m\bar{3}m$ | cube | *m* |
| 6 | 215.$P\bar{4}3m$ | (3) $\bar{4}3m$ (2) $\bar{4}2m$ (1) $\bar{4}2m$ (0) $\bar{4}3m$ | cube | *n* |
| 7 | 226.$Fm\bar{3}c$ | (3) 432 (2) $4/m$ (1) $\bar{4}2m$ (0) $m\bar{3}$ | cube | *m* |
| 8 | 207.$P432$ | (3) 432 (2) 422 (1) 422 (0) 432 | cube | *n* |
| 9 | 221.$Pm\bar{3}m$ | (3) $4/mmm$ (2) $mm2$ ($1_1$) $4/mmm$ ($1_2$) $3m$ ($0_1$) $m\bar{3}m$ ($0_2$) $m\bar{3}m$ | octa.* | *c* |
| 10 | 229.$Im\bar{3}m$ | (3) $\bar{4}2m$ (2) $mm2$ ($1_1$) $4/mmm$ ($1_2$) $\bar{3}m$ (0) $m\bar{3}m$ | tetra.* | *c* |
| 11 | 216.$F\bar{4}3m$ | (3) $\bar{4}3m$ (2) $mm2$ ($1_1$) $3m$ ($1_2$) $3m$ ($0_1$) $\bar{4}3m$ ($0_2$) $\bar{4}3m$ ($0_3$) $\bar{4}3m$ | rhomb. dodec. | *m* |
| 12 | 229.$Im\bar{3}m$ | (3) $\bar{3}m$ (2) $mm2$ ($1_1$) $4mm$ ($1_2$) $\bar{4}2m$ ($0_1$) $m\bar{3}m$ ($0_2$) $4/mmm$ | cube | *m* |
| 13 | 227.$Fd\bar{3}m$ | (3) $\bar{3}m$ (2) $mm2$ ($1_1$) $3m$ ($1_2$) $\bar{3}m$ ($0_1$) $\bar{4}3m$ ($0_2$) $\bar{4}3m$ | spec. rhomb.* | *c* |
| 14 | 166.$R\bar{3}m$ | (3) $\bar{3}m$ (2) $2/m$ (1) $2/m$ (0) $\bar{3}m$ | cube | *c,n* |
| 15 | 202.$Fm\bar{3}$ | (3) $\bar{3}m$ (2) $2/m$ (1) 3 ($0_1$) $m\bar{3}$ ($0_2$) 23 | rhomb. dodec. | *n* |
| 16 | 202.$Fm\bar{3}$ | (3) 23 (2) $2/m$ (1) $mm2$ ($0_1$) $m\bar{3}$ ($0_2$) $m\bar{3}$ | cube | *m* |
| 17 | 195.$P23$ | (3) 23 (2) 222 (1) 222 (0) 23 | cube | *n* |
| 18 | 219.$F\bar{4}3c$ | (3) 23 (2) $\bar{4}$ (1) $\bar{4}$ (0) 23 | cube | *n* |
| 19 | 209.$F432$ | (3) 432 (2) 222 (1) 3 ($0_1$) 23 ($0_2$) 432 | rhomb. dodec. | *n* |
| 20 | 209.$F432$ | (3) 23 (2) 222 (1) 4 ($0_1$) 432 ($0_2$) 432 | cube | *n* |
| 21 | 225.$Fm\bar{3}m$ | (3) $mmm$ (2) $m$ ($1_1$) $4mm$ ($1_2$) $3m$ ($1_2$) $3m$ ($0_1$) $m\bar{3}m$ ($0_2$) $m\bar{3}m$ ($0_3$) $\bar{4}3m$ | octa. | *m* |
| 22 | 215.$Pm\bar{4}3m$ | (3) $\bar{4}2m$ (2) $m$ ($1_1$) $3m$ ($1_2$) $3m$ ($1_3$) $\bar{4}\ 2m$ ($0_1$) $\bar{4}3m$ ($0_2$) $\bar{4}3m$ | octa. | *m* |
| 23 | 226.$Fm\bar{3}c$ | (3) $4/m$ (2) 2 ($1_1$) $\bar{4}2m$ ($1_2$) 3 ($0_1$) $m\bar{3}$ ($0_2$) 432 | octa. | *n* |
| 24 | 224.$Pn\bar{3}m$ | (3) 222 (2) $m$ ($1_1$) $\bar{3}m$ ($1_2$) $\bar{3}m$ ($1_3$) $\bar{4}2m$ (0) $\bar{4}3m$ | tetra. | *m* |
| 25 | 211.$I432$ | (3) 222 (2) 2 ($1_1$) 422 ($1_2$) 32 (0) 432 | tetra. | *n* |
| 26 | 217.$I\bar{4}3m$ | (3) $\bar{4}$ (2) $m$ ($1_1$) $\bar{4}2m$ ($1_2$) $3m$ (0) $\bar{4}3m$ | tetra. | *m* |
| 27 | 222.$Pn\bar{3}n$ | (3) $\bar{4}$ (2) 2 ($1_1$) 422 ($1_2$) $\bar{3}$ (0) 432 | tetra. | *n* |
| 28 | 207.$P432$ | (3) 422 (2) 2 ($1_1$) 422 ($1_2$) 3 ($0_1$) 432 ($0_2$) 432 | octa. | *n* |
| 29 | 223.$Pm\bar{3}n$ | (3) 32 (2) $m$ ($1_1$) $\bar{4}2m$ ($1_2$) $\bar{4}2m$ ($1_3$) $mm2$ ($0_1$) $mmm$ ($0_2$) $m\bar{3}$ | cube | *m* |
| 30 | 210.$F4_132$ | (3) 32 (2) 2 ($1_1$) 32 ($1_2$) 3 ($1_2$) 3($0_1$) 23 ($0_2$) 23 | spec. rhomb. | *n* |
| 31 | 211.$I432$ | (3) 32 (2) 2 ($1_1$) 222 ($1_2$) 4 ($0_1$) 422 ($0_2$) 432 | cube | *n* |
| 32 | 167.$R\bar{3}c$ | (3) 32 (2) $\bar{1}$ (1) 2 (0) $\bar{3}$ | cube | *n* |
| 33 | 155.$R32$ | (3) 32 (2) 2 (1) 2 (0) 32 | cube | *n* |
| 34 | 204.$Im\bar{3}$ | (3) $\bar{3}$ (2) $m$ ($1_1$) $mm2$ ($1_2$) $mm2$ ($0_1$) $mmm$ ($0_2$) $m\bar{3}$ | cube | *m* |
| 35 | 206.$Ia\bar{3}$ | (3) $\bar{3}$ (2) 2 (1) 2 (0) $\bar{3}$ | cube | *n* |
| 36 | 203.$Fd\bar{3}$ | (3) $\bar{3}$ (2) 2 ($1_1$) $\bar{3}$ ($1_2$) 3 ($0_1$) 23 ($0_2$) 23 | spec. rhomb. | *n* |
| 37 | 222.$Pn\bar{3}n$ | (3) $\bar{3}$ (2) 2 ($1_1$) $\bar{4}$ ($1_2$) 4 ($0_1$) 422 ($0_2$) 432 | cube | *n* |
| 38 | 148.$R\bar{3}$ | (3) $\bar{3}$ (2) $\bar{1}$ (1) $\bar{1}$ (0) $\bar{3}$ | cube | *n* |
| 39 | 167.$R\bar{3}c$ | (3) $\bar{3}$ (2) 2 (1) $\bar{1}$ (0) 32 | cube | *n* |
| 40 | 196.$F23$ | (3) 23 (2) 2 ($1_1$) 3 ($1_2$) 3 ($0_1$) 23 ($0_2$) 23 ($0_3$) 23 | rhomb. dodec. | *n* |
| 41 | 225.$Fm\bar{3}m$ | ($3_1$) $m\bar{3}m$ ($3_2$) $m\bar{3}m$ (2) $4mm$ (1) $mmm$ (0) $\bar{4}3m$ | cube–cube | *n* |
| 42 | 225.$Fm\bar{3}m$ | ($3_1$) $m\bar{3}m$ ($3_2$) $\bar{4}3m$ (2) $3m$ (1) $mmm$ (0) $m\bar{3}m$ | octa.–tetra.* | *c,n* |
| 43 | 202.$Fm\bar{3}$ | ($3_1$) $m\bar{3}$ ($3_2$) $m\bar{3}$ (2) $mm2$ (1) $2/m$ (0) 23 | cube–cube | *n* |
| 44 | 216.$F\bar{4}3m$ | ($3_1$) $\bar{4}3m$ ($3_2$) $\bar{4}3m$ (2) $mm2$ (1) $mm2$ ($0_1$) $\bar{4}3m$ ($0_2$) $\bar{4}3m$ | cube–cube | *n* |
| 45 | 202.$Fm\bar{3}$ | ($3_1$) $m\bar{3}$ ($3_2$) 23 (2) 3 (1) $2/m$ (0) $m\bar{3}$ | octa.–tetra. | *n* |
| 46 | 209.$F432$ | ($3_1$) 432 ($3_2$) 432 (2) 4 (1) 222 (0) 23 | cube–cube | *n* |
| 47 | 209.$F432$ | ($3_1$) 432 ($3_2$) 23 (2) 3 (1) 222 (0) 432 | octa.–tetra. | *n* |
| 48 | 139.$I4/mmm$ | ($3_1$) $4/mmm$ ($3_2$) $\bar{4}2m$ (2) $m$ ($1_1$) $mmm$ ($1_2$) $2/m$ (0) $4/mmm$ | octa.–tetra. | *c,n* |
| 49 | 223.$Pm\bar{3}n$ | ($3_1$) $\bar{4}2m$ ($3_2$) $\bar{4}2m$ (2) $m$ ($1_1$) $mmm$ ($1_2$) 32 (0) $m\bar{3}$ | tetra.–tetra. | *n* |
| 50 | 224.$Pn\bar{3}m$ | ($3_1$) $\bar{3}m$ ($3_2$) $\bar{3}m$ (2) $m$ ($1_1$) $mm2$ ($1_2$) 222 ($0_1$) $\bar{4}3m$ ($0_2$) $\bar{4}2m$ | cube–cube | *n* |
| 51 | 166.$R\bar{3}m$ | ($3_1$) $\bar{3}m$ ($3_2$) $\bar{3}m$ (2) $m$ ($1_1$) $2/m$ ($1_2$) $2/m$ (0) $3m$ | cube–cube | *n* |
| 52 | 196.$F23$ | ($3_1$) 23 ($3_2$) 23 (2) 2 (1) 2 ($0_1$) 23 ($0_2$) 23 | cube–cube | *n* |
| 53 | 69.$Fmmm$ | ($3_1$) $mmm$ ($3_2$) 222 (2) 1 ($1_1$) $2/m$ ($1_2$) $2/m$ ($1_3$) $2/m$ (0) $mmm$ | octa.–tetra. | *c,n* |
| 54 | 136.$P4_2/mmm$ | ($3_1$) $mmm$ ($3_2$) $\bar{4}$ (2) 1 ($1_1$) $2/m$ ($1_2$) $m$ (0) $mmm$ | octa.–tetra. | *n* |
| 55 | 134.$P4_2/mmm$ | ($3_1$) $\bar{4}2m$ ($3_2$) 222 (2) 1 ($1_1$) $2/m$ ($1_2$) $2/m$ ($1_3$) 222 (0) $\bar{4}2m$ | octa.–tetra. | *n* |
| 56 | 121.$I\bar{4}2m$ | ($3_1$) $\bar{4}2m$ ($3_2$) $\bar{4}$ (2) 1 ($1_1$) $m$ ($1_2$) 222 (0) $\bar{4}2m$ | octa.–tetra. | *n* |
| 57 | 128.$P4/mnc$ | ($3_1$) $4/m$ ($3_2$) 222 (2) 1 ($1_1$) $2/m$ ($1_2$) 2 (0) $4/m$ | octa.–tetra. | *n* |
| 58 | 87.$I4/m$ | ($3_1$) $4/m$ ($3_2$) $\bar{4}$ (2) 1 ($1_1$) $2/m$ ($1_2$) $\bar{1}$ (0) $4/m$ | octa.–tetra. | *n* |
| 59 | 208.$P4_232$ | ($3_1$) 222 ($3_2$) 222 (2) 1 ($1_1$) 32 ($1_2$) 32 ($1_3$) 222 (0) 23 | tetra.–tetra. | *n* |
| 60 | 97.$I422$ | ($3_1$) 222 ($3_2$) 422 (2) 1 ($1_1$) 222 ($1_2$) 2 (0) 422 | octa.–tetra. | *n* |
| 61 | 218.$P\bar{4}3n$ | ($3_1$) $\bar{4}$ ($3_2$) $\bar{4}$ (2) 1 ($1_1$) 222 ($1_2$) 3 (0) 23 | tetra.–tetra. | *n* |
| 62 | 126.$P4/nnc$ | ($3_1$) $\bar{4}$ ($3_2$) 422 (2) 1 ($1_1$) 222 ($1_2$) $\bar{1}$ (0) 422 | octa.–tetra. | *n* |
| 63 | 208.$P4_232$ | ($3_1$) 32 ($3_2$) 32 (2) 1 ($1_1$) 222 ($1_2$) 222 ($1_3$) 2 ($0_1$) 222 ($0_2$) 23 | cube–cube | *n* |
| 64 | 155.$R32$ | ($3_1$) 32 ($3_2$) 32 (2) 1 ($1_1$) 2 ($1_2$) 2 (0) 3 | cube–cube | *n* |
| 65 | 228.$Fd\bar{3}c$ | ($3_1$) 32 ($3_2$) $\bar{3}$ (2) 1 ($1_1$) 2 ($1_2$) 2 ($0_1$) $\bar{4}$ ($0_2$) 23 | cube–cube | *n* |
| 66 | 167.$R\bar{3}c$ | ($3_1$) 32 ($3_2$)$\bar{3}$ (2) 1 ($1_1$) 2 ($1_2$) $\bar{1}$ (0) 3 | cube–cube | *n* |
| 67 | 201.$Pn\bar{3}$ | ($3_1$) $\bar{3}$ ($3_2$) $\bar{3}$ (2) 1 ($1_1$) 2 ($1_2$) 2 ($0_1$) 222 ($0_2$) 23 | cube–cube | *n* |
| 68 | 205.$Pa\bar{3}$ | ($3_1$) $\bar{3}$ ($3_2$) $\bar{3}$ (2) 1 (1) 1 (0) 3 | cube–cube | *c,n* |
| 69 | 148.$R\bar{3}$ | ($3_1$) $\bar{3}$ ($3_2$) $\bar{3}$ (2) 1 ($1_1$) $\bar{1}$ ($1_2$) $\bar{1}$ (0) 3 | cube–cube | *n* |
| 70 | 160.$R3m$ | (3) $3m$ (2) $m$ (1) $m$ (0) $3m$ | cube | *n* |
| 71 | 217.$I\bar{4}3m$ | (3) $3m$ (2) $m$ ($1_1$) $mm2$ ($1_2$) $\bar{4}$ ($0_1$) $\bar{4}3m$ ($0_2$) $\bar{4}2m$ | cube | *n* |
| 72 | 200.$Pm\bar{3}$ | (3) $mmm$ (2) $m$ ($1_1$) $mmm$ ($1_2$) 3 ($0_1$) $m\ \bar{3}$ ($0_2$) $m\bar{3}$ | octa. | *n* |
| 73 | 204.$Im\bar{3}$ | (3) $mm2$ (2) $m$ ($1_1$) $mmm$ ($1_2$) $\bar{3}$ (0) $m\bar{3}$ | tetra. | *n* |
| 74 | 226.$Fm\bar{3}c$ | (3) $\bar{4}2m$ (2) $m$ ($1_1$) $4/m$ ($1_2$) 3 ($0_1$) 432 ($0_2$) $m\bar{3}$ | octa. | *n* |
| 75 | 202.$Fm\bar{3}$ | (3) $2/m$ (2) 1 ($1_1$) $mm2$ ($1_2$) 3 ($1_3$) 3 ($0_1$) $m\bar{3}$ ($0_2$) $m\bar{3}$ ($0_3$) 23 | octa. | *n* |
| 76 | 195.$P23$ | (3) 222 (2) 1 ($1_1$) 222 ($1_2$) 3 ($1_3$) 3 ($0_2$) 23 ($0_1$) 23 | octa. | *n* |
| 77 | 197.$I23$ | (3) 2 (2) 1 ($1_1$) 222 ($1_2$) 3 (0) 23 | tetra. | *n* |
| 78 | 228.$Fd\bar{3}c$ | (3) 2 (2) 1 ($1_1$) 32 ($1_2$) $\bar{3}$ ($1_3$) $\bar{4}$ (0) 23 | tetra. | *n* |
| 79 | 201.$Pn\bar{3}$ | (3) 2 (2) 1 ($1_1$) 222 ($1_2$) $\bar{3}$ ($1_3$) $\bar{3}$ (0) 23 | tetra | *n* |
| 80 | 205.$Pa\bar{3}$ | (3) $\bar{3}$ (2) 1 (1) 1 (0) $\bar{3}$ | cov. rhomb.* | *c,n,* |
| 81 | 209.$F432$ | (3) 222 (2) 1 ($1_1$) 4 ($1_2$) 3 ($1_3$) 3 ($0_1$) 432 ($0_2$) 432 ($0_3$) 23 | octa. | *n* |
| 82 | 219.$F\bar{4}3c$ | (3) $\bar{4}$ (2) 1 ($1_1$) 3 ($1_2$) 3 ($1_3$) $\bar{4}$ ($0_1$) 23 ($0_2$) 23 | octa. | *n* |
| 83 | 205.$Pa\bar{3}$ | (3) $\bar{3}$ (2) 1 ($1_1$) 1 ($1_2$) 3 ($0_1$) $\bar{3}$ ($0_2$) 3 | rhomb. dodec. | *n* |
| 84 | 161.$R3c$ | (3) 3 (2) 1 (1) 1 (0) 3 | cube | *n* |
| 85 | 197.$I23$ | (3) 3 (2) 1 ($1_1$) 2 ($1_2$) 2 ($0_1$) 222 ($0_2$) 23 | cube | *n* |
| 86 | 205.$Pa\bar{3}$ | (3) 3 (2) 1 (1) 1 ($0_1$) $\bar{3}$ ($0_2$) $\bar{3}$ | cube | *n* |
| 87 | 218.$P\bar{4}3n$ | (3) 3 (2) 1 ($1_1$) 2 ($1_2$) $\bar{4}$ ($1_3$) $\bar{4}$ ($0_1$) 222 ($0_2$) 23 | cube | *n* |
| 88 | 146.$R3$ | (3) 3 (2) 1 (1) 1 (0) 3 | cube | *n* |

† Abreviations: rhomb. dodec., rhombic dodecahedron; octa., octahedron; tetra., tetrahedron; spec. rhomb., special rhombohedron; cov. rhomb., covered rhombohedron.

4, $m_{12}(D) = 3$ and $m_{23}(D) = 4$, it follows that the corresponding tiling is generated by a characteristic simplex $C$ with Coxeter diagram (see Coxeter & Moser, 1980):




The group $\Gamma$ is generated by reflections as the side planes of $C$ (and thus the convex realization of the tiling is unique). This is because $s_i(D) = D$ for all $i \in \{0,1,2,3\}$. So, in fact, $\Gamma$ is a group of type 221. $Pm\bar{3}m$ (see Hahn, 1983). The corners of $C$ are the centers of the '[3], [2], [1] and [0]' stabilizers listed in Table 4. From this tiling, where $\Gamma = \mathrm{Aut}(\mathcal{T})$, the other 42 equivariant cubic tilings can be derived by symmetry breaking (see Fig. 10). A detailed study of the cubic case can be found in Molnár (1992).

Choosing 4, 3, 3 or 4, 3, 5 for the values of $m_{01}(D)$, $m_{12}(D)$ and $m_{23}(D)$ leads to spherical or hyperbolic face-transitive tilings, respectively (see Coxeter, 1954, 1956).

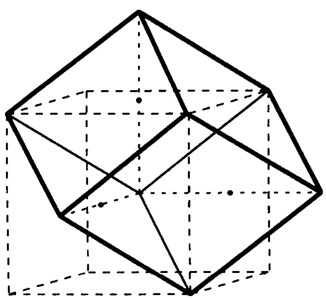


Fig. 8. A special rhombohedron fitted into the cubic tiling. Four of the vertices lie in cube centers and the other four lie on cube corners. The tiling can also be obtained from the tiling by rhombic dodecahedra by evenly splitting each rhombic dodecahedron into four special rhombohedra.

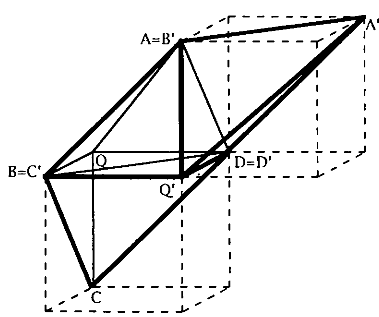



Fig. 9. A fundamental region $F = ABCDQ$ of the tiling corresponding to symbol no. 80 in Table 1. The set $F' = A'B'C'D'Q'$ is a copy of $F$ produced by a rotatory inversion of type $\bar{3}$ about the point $D = D'$.

Table 3. *The five Delaney symbols whose symmetry skeletons are compatible with a crystallographic space group but which do not give rise to face-transitive tilings of Euclidean space*

Here we list all five Delaney symbols that are both minimal and imply a symmetry skeleton that is compatible with at least one crystallographic space-group type (listed after each symbol) but which we claim *do not* give rise to a periodic tiling of Euclidean space. The symbols are encoded as in Table 1. Note that nos. 90, 91 and 92 all have the same Delaney graph and only differ in the values of the functions $m_{ij}$. In fact, all three symbols give rise to groups that are conjugate by isometries of the hyperbolic space $\mathbb{H}^3$.

| No. | Delaney symbol |
|---|---|
| 89 | $D$ = 6 : 2 4 6, 6 3 5, 2 6 5, 4 3 6 $m$ = 3, 4 3, 3 |
| 90 | $D$ = 10 : 2 4 6 8 10, 10 3 5 7 9, 2 6 5 10 9, 6 5 4 10 9 $m$ = 5, 3 3 3, 6 4 4 |
| 91 | $D$ = 10 : 2 4 6 8 10, 10 3 5 7 9, 2 6 5 10 9, 6 5 4 10 9 $m$ = 5, 3 3 4, 6 3 3 |
| 92 | $D$ = 10 : 2 4 6 8 10, 10 3 5 7 9, 2 6 5 10 9, 8 7 6 5 10 $m$ = 5, 3 3 4, 6 3 3 |
| 93 | $D$ = 12 : 2 4 6 8 10 12, 6 3 5 12 9 11, 2 7 8 11 12 10, 7 8 9 10 11 12 $m$ = 3 3, 6 3, 4 3 3 |

Table 4. *Stabilizer groups and Euclidean space groups for Delaney symbols in Table* 3

For each of the five Delaney symbols in Table 3, we list the encoded stabilizer groups and all Euclidean space groups that have compatible symmetry skeletons. Furthermore, we indicate in which space the encoded tiling can actually be realized and, in most cases, the name of the symmetry group.

| No. | Stabilizer groups | Compatible Euclidean space groups | Realization space, symmetry group | References |
|---|---|---|---|---|
| 89 | (3) 32 (2) 2 (1) 2 (0) 32 | 212.$P4_332$, 213.$P4_132$ | $\mathbb{S}^3$, $K_{32}$ | Zhuk (1983) |
| 90 | (3) 23 (2) 2 $(1_1)$ 222 $(1_2)$ 4 $(0_1)$ 432 $(0_2)$ 432 | 209.$F432$ | $\mathbb{H}^3, \Gamma(q, 6), q = 4$ | Molnár (1993) |
| 91 | (3) 432 (2) 2 $(1_1)$ 222 $(1_2)$ 3 $(0_1)$ 23 $(0_2)$ 432 | 209.F432 | $\mathbb{H}^3, \Gamma(q, 6), q = 4$ | |
| 92 | (3) 432 (2) 2 $(1_1)$ 222 $(1_2)$ 3 $(0_1)$ 432 $(0_2)$ 23 | 209.F432 | $\mathbb{H}^3, \Gamma(q, 6), q = 4$ | |
| 93 | (3) 222 (2) 1 $(1_1)$ 2 $(1_2)$ 3 (0) 23 | 197.$I23$ | $\mathbb{S}^3$ | Dunbar (1988, p.93) |

*Example* 2. Let $(\mathcal{D};m)$ be the Delaney symbol no. 14 in Table 1. Consider Fig. 11. The four vertices $D_1$, $D_2$, $D_3$ and $D_4$ of the Delaney graph give rise to four simplices $D_1$, $D_2$, $D_3$ and $D_4$, glued together as implied by the equations $s_0(D_1) = D_2$, $s_1(D_2) = D_3$ and $s_0(D_3) = D_4$.

The equations $s_1(D_1) = D_1$ and $s_1(D_4) = D_4$, together with $v_{01}(D) = m_{01}(D)/r_{01}(D) = 4/4 = 1$ for $D \in \mathcal{D}$, imply a plane reflection in the face $f_1(D_1) \cup f_1(D_4)$, where $f_k(C)$ denotes the $k$ face of the chamber $C$. The equations $s_2(D_1) = D_1$ and $s_2(D_2) = D_2$, together with $v_{02}(D) = m_{02}(D)/r_{02}(D) = 2/2 = 1$ for $D \in \{D_1, D_2\}$, imply a plane reflection in $f_2(D_1) \cup f_2(D_2)$. The equation $s_2(D_3) = D_4$ with $v_{02}(D) = m_{02}(D)/r_{02}(D)$

$= 2/1 = 2$ for $D \in \{D_3, D_4\}$ implies a half-turn around the axis $e_{13}(D_2)$, where $e_{ij}(C)$ denotes the edge joining the $i$ and $j$ vertex of a chamber $C$. Similarly, the equations $s_3(D_1) = D_4$ and $s_3(D_2) = D_3$ indicate a half-turn $f_3(D_1) \cup f_3(D_2) \rightarrow f_3(D_3) \cup f_3(D_4)$ about $e_{02}(D_2)$. The rotational order around the edge $e_{03}(D_1)$

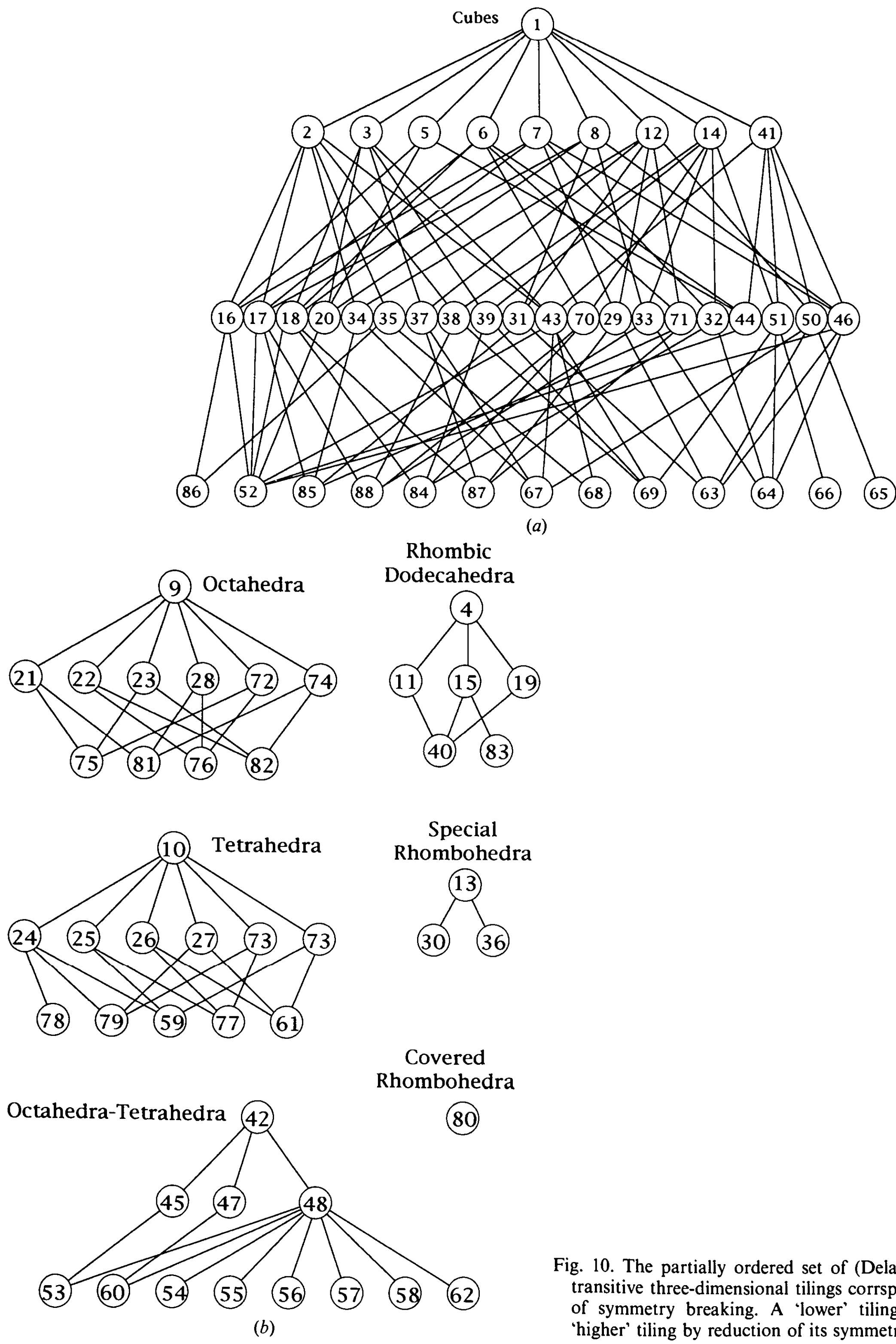



Fig. 10. The partially ordered set of (Delaney symbols of) face-transitive three-dimensional tilings corrsponding to the relation of symmetry breaking. A 'lower' tiling is obtained from a 'higher' tiling by reduction of its symmetry group.

is $v_{12}(D_1) = m_{12}(D_1)/r_{12}(D_1) = 3/1 = 3$. For the two equivalent edges $e_{03}(D_2)$ and $e_{03}(D_4)$, this order is $v_{12}(D_3) = 3/3 = 1$.

These considerations lead us to a group of type 166. $R\bar{3}m$ and a tiling $\mathcal{T}$, which is topologically a cube tiling. As indicated by the letter $R$, the symmetry group possesses a rhombohedral lattice and thus a free stretching parameter. A 'general' convex rhombohedron (*i.e.* one different from the cube and from the 'special rhombohedron' depicted in Fig. 8, having six face angles of 120° and six face angles of 60°) can be used to produce the tiling. A nonconvex realization is always possible by 'nicely' bending the face part $f_3(D_1) \cup f_3(D_2)$.

*Example* 3. The most interesting tiling $(\mathcal{T},\Gamma)$ found in this investigation is perhaps the one encoded by the Delaney symbol $(\mathcal{D};m)$ listed as no. 80 in Table 1. We call the tiles *covered rhombohedra* or *triangle dodecahedra* because the faces of neighboring rhombs partly cross each other along common regular triangles (see Fig. 9). This tiling can be found in the work of Grünbaum & Shephard (1980). The fundamental domain $F = ABCDQ$ in this case consists of 12 simplices glued together, in the way the Delaney symbol prescribes. The operation of $s_3$ implies a rotatory inversion of type $\bar{3}$ with fixed point $D = D'$, which produces the crossing image $F' = A'B'C'D'$. The covered rhombohedron has $Q$ as its center with stabilizer of type $\bar{3}$ and of order six. Hence, six images of $F$ around $Q$ make up the whole tile. The construction leads to a space group of type 205. $Pa\bar{3}$.

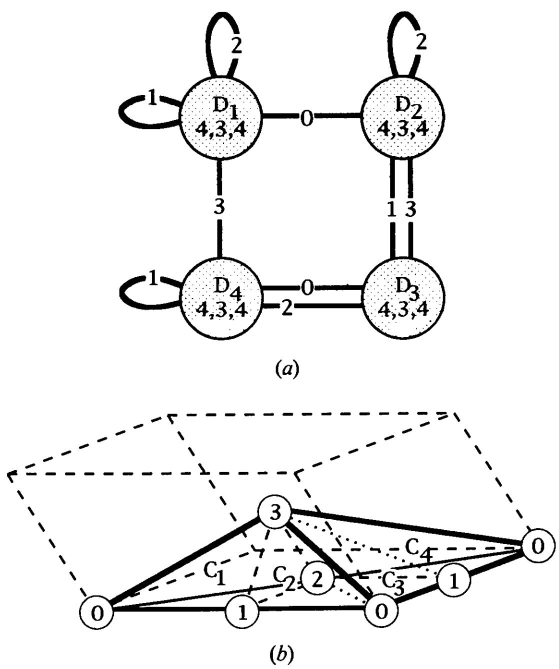



Fig. 11. (*a*) The Delaney graph corresponding to symbol no. 14 in Table 1. The four circles $D_1$, $D_2$, $D_3$ and $D_4$ represent the vertices of the graph and each contains three numbers indicating the values of $m_{01}$, $m_{12}$ and $m_{23}$. Each line or curve labeled $i$ represents an $i$ edge ($i \in \{0,1,2,3\}$). (*b*) Here, we depict the union of four chambers $C_1$, $C_2$, $C_3$ and $C_4$, corresponding to the four vertices of the Delaney graph. Small labeled circles indicate the 0, 1, 2 and 3 vertices of the four chambers. The four chambers make up a fundamental region (solid lines) for the corresponding tiling. Two dotted lines indicate half-turn axes.

*Note added in proof*: Since this paper was submitted, Olaf Delgado Friedrichs has developed computer programs that – by systematically breaking non-translation symmetry – established automatically that the symbols in Tables 3 and 4 do not have Euclidean realizations.

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